Significance of Momentum wave function.

We have represented the wave function as the superposition of infinite number of plane waves ~~a(p)~~ as $a(p)$ as

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(p)\, e^{\frac{i}{\hbar}(px - Et)}\, dp$$

In 3D

$$\psi(r,t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} a(p)\, e^{\frac{i}{\hbar}(p \cdot r - Et)}\, dp$$

$a(p)$ is the momentum wavefunction which is also the amplitude of the particular plane wave corresponding to a definite value of p.

Now we wish to find what p is represented in $a(p)$ or $e^{\frac{i}{\hbar}(p\cdot r)}$

To have this we operate momentum operator $-i\hbar\nabla$ on $e^{\frac{i}{\hbar}p\cdot r}$

$$-i\hbar\nabla\left(e^{\frac{i}{\hbar}p\cdot r}\right) = (-i\hbar)\left(\frac{i}{\hbar}p\right) e^{\frac{i}{\hbar}p\cdot r}$$

$$\Rightarrow -i\hbar\nabla\left(e^{\frac{i}{\hbar}p\cdot r}\right) = p\, e^{\frac{i}{\hbar}(p\cdot r)}$$

Which is an eigen value equation for the momentum operator, indicating that $e^{\frac{i}{\hbar}p\cdot r}$ is an eigen function of the momentum operator.

with eigen value P.

We conclude therefore that the possible momenta are given by p and the momentum eigen-functions will depend on the value of P.

Any one of ~~them can~~ them can be represented as $a(p)\, e^{\frac{i}{\hbar} P.r}$. Therefore the momentum ~~eigenfunction~~ wavefunction $a(p)$ that we have defined are the amplitude of all possible momentum eigen functions, that superpose on the wave function $\psi(r,t)$.

The Commutator

The commutator of two operators $\hat{A}$ & $\hat{B}$ is represented by the relation $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$.

If the commutator of two operators is non zero, Then the ~~two operators~~ eigen values corresponding to the two operators can not be measured simultaneously. The nonzero commutator of two operators means that They are following the uncertainty relation of Heisenberg.